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Symmetry and Composite Particles†

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Abstract

The description of systems with internal structure evident only in the presence of operators other than P_μ and $M_{\mu\nu}$ defined on the space S of the relativistic states is explored. The assumption is made that there must be defined *one* operator of this kind, namely $\mu \equiv \lim_{c\to\infty} P_0/c$. The individual particle variables of a composite are eliminated and a special kind of 'composite particle representation' is introduced. The nonrelativistic composite (an isolated molecule) is described by a single value of the Galilei invariant μ and many values of the other two Casimir operators C_1 and C_2, and it is this direct sum structure which is sought as the contracted limit of a relativistic composite particle with all its bound state levels. The internal energy operator H_0 arises as the contraction of $(M-\mu)c^2$, and the correct nonrelativistic Hamiltonian of the composite is shown to be given by $H = H_0 + \mathbf{P}^2/2\mu$. As an application of these ideas, our previous results (compare Fleischman & Roman, 1967) concerning the nonrelativistic limit of the SU_3 commutator structure are rederived in a simple manner. Throughout the paper, comparison is made with, and ramificational remarks are proferred on the extensive related literature.

1. *Introduction*

The space-time symmetries of a relativistic particle are well known. Wigner (1939) showed that the Hilbert space accommodating all states of a free relativistic particle carries a unitary irreducible representation (I.R.) of the Poincaré group $\mathscr{P}$. Given a state ψ, all other states ψ' of the particle can be reached via the unitary operator

$$U(\Lambda, a) = \exp i(\tfrac{1}{2}\Lambda^{\mu\nu} M_{\mu\nu} + a_\lambda P^\lambda)$$

i.e.,

$$\psi'(x_\mu') = U(\Lambda, a)\,\psi(x_\mu)$$

Here $x_\mu' = \Lambda_\mu{}^\nu x_\nu + a^\nu$, and $M_{\mu\nu}$ and P^λ are the ten hermitian generators of the Poincaré group. The I.R. associated with the particle is fully characterized by the eigenvalues of the two Casimir operators $P_\mu P^\mu = -M^2c^2$ and $W_\mu W^\mu = -M^2c^2 s(s+1)$, where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda} M^{\nu\sigma} P^\lambda$.

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The mass and spin parameters M' and s' uniquely characterize a free relativistic particle. The possible relativistic wave equations for a free particle were obtained by Bargmann & Wigner (1946).

We note here that a *composite particle* is described via *reducible* representations of $\mathscr{P}$, and only the ground state is associated† with an I.R.

Now, it is well known that isolated hadrons phenomenologically possess many features of 'bound states' of a composite system. This is exemplified by the structure of SU_3 multiplets with fixed spin and parity, or by the higher mass 'recurrences' of hadrons with given internal quantum numbers and parity at higher spin values. While several composite particle models (such as the quark model) have been suggested, in the absence of a detailed dynamical law these models cannot be seriously considered as a description of the dynamics.

In view of the absence of such a dynamical law, we have explored and we present in this paper a problem which is closely related to dynamics. We shall discuss the description of a system with internal structure evident only in the presence of operators other than P_μ and $M_{\mu\nu}$ defined on the space S of the relativistic states.‡ We have been led to the assumption that there must be defined *one* particular operator μ of this kind.§

Some aspects of our approach overlap with those of Foldy's work (Foldy, 1961) on the description of the relativistic interaction of a *fixed number* of particles, without the introduction of the field concept. The notion of relativistic invariance used by Foldy is an extension of that given by Dirac (1949) and used by Bakamjian & Thomas (1953). This concept of invariance has been amplified by Currie *et al.* (1964), who also prove a contradiction between the realization of a nontrivial interaction via Poincaré canonical transformations on one hand and the covariant transformation of the position operator of a particle on the other hand.‖

† We are indebted to H. P. Dürr (Max Planck Institute and University, Munich) for an interesting discussion concerning the decay of relativistic states and the impossibility of describing them by I.R.'s of $\mathscr{P}$. One cannot mathematically 'isolate' a relativistic system in the way that is possible for a non-relativistic system. We shall illuminate this point further in the text. See also the discussion of the hydrogen atom by Newton & Wigner (1949).

‡ Of course, the usual internal symmetry operators **I**, Y, B, Q as well as the discrete symmetry operators P, C, T are also defined on S.

§ The precise definition of μ is given in equation (2.1.4).

‖ The notion of position operators q_k at a given time for localizable elementary relativistic *systems* has been introduced and explored by Newton, T. D. & Wigner, E. P. (1949).

We, however, are not concerned with the position operator or the criterion of localizability of a particle, which, as Newton & Wigner (1949) pointed out, are aspects that 'get lost' in the group theoretic fundamental description. Indeed, we take the point of view regarding composite systems that the individual particles are really 'lost', and that the dynamical variables describing a composite may not even include these entities. For example, the electrons may not 'really' exist 'inside' the composite atom any more than the photons do.

We thus extend the ideas of Foldy (1961) of dealing with the relativistic requirements *only*, and not with 'further extraneous conditions which may nevertheless be necessary in order to yield a theory which is physically satisfactory'. Foldy himself has eliminated manifest covariance (which is not needed), pair creation (which is not wanted in his approximation), separability of the interaction (which he could not achieve) and causality (which is only mentioned). We eliminate the individual particle variables of the composite as well, and we shall deal with a special kind of *composite particle representation* of the space-time symmetry group. The question of interactions must be postponed until we understand the particles themselves. Of course, these additional properties should not be *excluded* by our formalism.

The suggested composite particle representation is most easily understood when examining the *nonrelativistic aspect* of the problem considered by Foldy (1961) and Currie, *et al.* (1964). The space-time symmetry group of nonrelativistic quantum mechanics is embodied in a special representation† of the inhomogeneous Galilei group $\mathscr{G}$. The physical interpretation of these representations has not yet been fully explored, even though an excellent start has been made in the papers by Hammermesh (1960) and Levy-Leblond (1963). It appears that a great deal more can be said with very little mathematical sophistication or detailed calculation. This study constitutes Section 2 of our paper.

Levy-Leblond (1963) emphasizes the meaning of the Galilean mass superselection rule: it shows that nonrelativistic unstable elementary particles cannot exist. We add to this that every state in an I.R. of $\mathscr{G}$ is a stationary state in its own restframe; hence it must be stable. Thus, nonrelativistic composite particles are also stable.

† The special representations of $\mathscr{G}$ which are capable of describing particles are called from the *technical* point of view, unitary nontrivial projective representations of the group of Galilean transformations on the space-time manifold. These representations are not equivalent to the true representations, and are actually central extensions of the latter. See Levy-Leblond (1963), Bargmann (1954), and Voison (1965b).

This fact is of profound importance. The concept of a nonrelativistic composite particle is the generalization of an isolated molecule. The key word is *isolated*, for no physical system can actually be isolated. However, the I.R.'s of $\mathscr{G}$ describe single entities with a well defined position operator $\mathbf{R}$ which is canonically conjugate to the total momentum operator $\mathbf{P}$, and with internal properties, namely the internal angular momentum operator $\mathbf{l}$ and the internal energy operator H_0. If one then turns on an interaction with the rest of the Universe, this isolated system can change its 'particle state' of motion ($\mathbf{R}$ and $\mathbf{P}$) through elastic processes, and it can also undergo internal changes (in $\mathbf{l}$ and H_0) through inelastic processes.

Even though one can never truly isolate a system, the properties (attributes) that the system can possess are described by way of extrapolation. In nonrelativistic quantum mechanics the isolated interacting system has a well defined mathematical model (in terms of a potential). This is not true in the relativistic domain, and hence one can never decouple the object of consideration from the outside, except for a free particle.

In Section 3 we construct a relativistic system which, in the limit when the light velocity c becomes infinite, has the vital properties of a composite nonrelativistic particle. This limit of an I.R. of $\mathscr{P}$ was first explored by Inönü & Wigner (1953), and subsequently they also presented (Inönü & Wigner, 1952) the first discussion of the I.R.'s of $\mathscr{G}$. A comprehensive mathematical study of $\mathscr{G}$ and its representations is contained in Bargmann's extensive treatment (Bargmann, 1954) of the ray representations of Lie groups. One crucial aspect of the particle I.R.'s of $\mathscr{G}$ is Bargmann's mass superselection rule which we already mentioned.

In a previous paper (Fleischman & Roman, 1967) we used the *known* relationship (Fleischman & Nagel, 1966) between the space-time inhomogeneous Galilei group of nonrelativistic quantum mechanics on the one hand and the *exact* internal symmetry group on the other, as a tool to discover facts concerning the corresponding *unknown* relationship for relativistic particles. We found (Fleischman & Roman, 1967) that a set of hadrons with given fixed spin and parity had as its nonrelativistic limit a set of states similar to those of an atom (or molecule), with internal energy eigenvalues E_0 obeying the Gell-Mann-Okubo (GMO) mass splitting formula

$$E_0(t, y) = A + By + C[t(t+1) + y^2/4]$$

This result did not depend upon assuming a quark model or any other composite system of interacting relativistic particles. Nor did we

assume a splitting of the interaction into a very strong and a medium strong part. We did, however, make the assumption that P_μ was the sum of two vectors $P_\mu^{(1)}$ and $P_\mu^{(2)}$ which behave differently in the limit $c \to \infty$. In particular we assumed that $[P_\mu^{(1)}, P_\nu^{(1)}] = 0$, and that $P_\mu^{(1)}$ and $[P_\mu, P_\nu^{(1)}]$ tend to zero when $c \to \infty$. This defined four operators $P_\mu^{(1)}$ (in addition to the Poincaré generators P_μ and $M_{\mu\nu}$) which act on the relativistic state space S. We also assumed that each hadron, separately, was described by means of an I.R. of $\mathscr{P}$.

At the end of Section 3 we shall show that our previous results (Fleischman & Roman, 1967) can be recovered without making any of these detailed assumptions on P_μ, using only the assumption of Section 2, namely that the Galilean mass operator $\mu = \lim_{c\to\infty} P_0/c$ is defined on the space S of relativistic states. In this respect we remark that the interaction studies of Foldy (1961) and Currie *et al.* (1964) naturally assume that many more operators (namely, the individual particle dynamical variables) are defined (in addition to P_μ and $M_{\mu\nu}$) on the space S. Even if one removes this feature from Foldy's work (Foldy, 1961), there still remains the fact that he has assumed P_0 and M_{0j} to possess expansions in powers of c^{-1} and that therefore many additional operators are defined on S. In contrast, our construct of Section 2 makes a much weaker assumption, namely that of μ alone being an additional operator on S.

2. *Non-Relativistic Composite Particles*

2.1. *The Contraction $\mathscr{P} \to \mathscr{G}$*

Let us begin our construction of a nonrelativistic particle description with a unitary representation of the Poincaré group $\mathscr{P}$. The commutation relations of the Hermitian generators P_μ and $M_{\mu\nu}$ are

$$[P_\mu, P_\nu] = 0 \tag{2.1.1a}$$

$$[M_{\mu\nu}, P_\lambda] = i(g_{\mu\lambda} P_\nu - g_{\nu\lambda} P_\mu) \tag{2.1.1b}$$

$$[M_{\mu\nu}, M_{\sigma\lambda}] = i(g_{\mu\sigma} M_{\nu\lambda} + g_{\nu\lambda} M_{\mu\sigma} - g_{\mu\lambda} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\lambda}) \tag{2.1.1c}$$

Our metric is $g_{00} = -1$, $g_{kk} = +1$ and we take $\hbar = 1$. The positive timelike representations are suitable to describe particles, whence the mass operator is given by the Casimir operator

$$P_\mu P^\mu = -M^2 c^2 \tag{2.1.2}$$

The contraction procedure is defined by keeping the generators P and $J_i \equiv \epsilon_{0ijk} M^{jk}$ $(i = 1, 2, 3)$ of the Euclidean subgroup of $\mathscr{P}$

fixed, while P_0 and M_{0i} are taken to be of order c. The pure Lorentz transformations thus go to the pure Galilean velocity transformations generated by K_i, where

$$K_i \equiv \lim_{c\to\infty} M_{0i}/c \tag{2.1.3}$$

The mass operator μ is defined by

$$\mu \equiv \lim_{c\to\infty} P_0/c \tag{2.1.4}$$

Thus we have defined the set Γ of ten generators

$$\Gamma = \{\mathbf{P}, \mathbf{J}, \mathbf{K}, \mu\} \tag{2.1.5}$$

as the limit of the ten generators $\mathscr{P} = \{P_\mu, M_{\mu\nu}\}$. The set Γ is a closed Lie algebra, and we have

$$\left.\begin{aligned} [P_i, P_j] &= [K_i, K_j] = 0 \\ [\mu, P_i] &= [\mu, J_i] = [\mu, K_i] = 0 \\ [K_i, P_j] &= i\delta_{ij}\mu \\ [J_i, V_j] &= i\epsilon_{ijk} V_k \end{aligned}\right\} \tag{2.1.6}$$

where we have used the shorthand

$$\mathbf{V} \equiv \{\mathbf{J}, \mathbf{P}, \mathbf{K}\} \tag{2.1.6a}$$

The above contraction process, however, has not defined the vital generator of implicit temporal evolution, i.e., the Hamiltonian H which (in the Schrödinger picture, for example) determines the dynamical development of the wave function through the time dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{2.1.7}$$

The representations studied by Hammermesh (1960) and by Levy-Leblond (1963) are essentially those for which H is the limit of the relativistic kinetic energy operator T, i.e.,

$$T \equiv c(P_0 - \sqrt{[-P_\mu P^\mu]}) \to \mathbf{P}^2/2\mu \equiv H \tag{2.1.8}$$

The operator H so defined commutes with all the operators of Γ except $\mathbf{K}$ and we find

$$[H, K_j] = -iP_j \tag{2.1.9}$$

By having chosen the representations for which H is given by (2.1.8), *the above authors have eliminated all composite particles,* and obtain the Schrödinger equation for a free particle.

To see more explicitly what this means, let us consider the whole algebra, *including* H. One finds that there are three operators which commute with all operators of $\mathscr{G}$. These are

$$\mu \tag{2.1.10a}$$

$$C_1 \equiv (\mu \mathbf{J} + \mathbf{K} \times \mathbf{P})^2 \tag{2.1.10b}$$

$$C_2 \equiv 2\mu H - \mathbf{P}^2 \tag{2.1.10c}$$

An I.R. is characterized by three parameters, the eigenvalues of μ, C_1, and C_2. *Hammermesh and Levy-Leblond have chosen* $C_2 = 0$, the former without discussion and the latter with a number of comments, which we shall elaborate on. But first we wish to recall the usual treatment of a nonrelativistic molecule (or atom).

2.2. *The isolated nonrelativistic molecule*

The Hamiltonian H of the isolated molecule is a given function of the individual particle masses $m^{(\alpha)}$, positions $\mathbf{r}^{(\alpha)}$, canonical momenta $\mathbf{p}^{(\alpha)}$, and spins $\boldsymbol{\sigma}^{(\alpha)}$. One then defines μ, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ by setting

$$\left.\begin{aligned} \mu &\equiv \sum m^{(\alpha)} \\ \mathbf{P} &\equiv \sum \mathbf{p}^{(\alpha)} \\ \mathbf{J} &\equiv \sum (\mathbf{r}^{(\alpha)} \times \mathbf{p}^{(\alpha)} + \boldsymbol{\sigma}^{(\alpha)}) \\ \mathbf{K} &\equiv t\mathbf{P} - \mu \mathbf{R} \end{aligned}\right\} \tag{2.2.1}$$

where we use the symbol $\mathbf{R}$ for the center-of-mass position operator,

$$\mathbf{R} = \mu^{-1} \sum m^{(\alpha)} \mathbf{r}^{(\alpha)} \tag{2.2.1a}$$

We note here that the structure of $\mathbf{K}$ is derived (Fleischman & Nagel, 1966) from the requirement that it transform under rotations like a vector and that $U(\mathbf{v}) \equiv \exp(i\mathbf{K}.\mathbf{v})$ produce the transformations $\mathbf{r}^{(\alpha)} \to \mathbf{r}^{(\alpha)} + \mathbf{v}t$, $\mathbf{p}^{(\alpha)} \to \mathbf{p}^{(\alpha)} + m^{(\alpha)}\mathbf{v}$, $\boldsymbol{\sigma}^{(\alpha)} \to \boldsymbol{\sigma}^{(\alpha)}$.

The operators defined by (2.2.1) satisfy the commutation relations (2.1.6) of $\mathscr{G}$, and H is chosen to be a function of the individual particle variables which commutes with μ, $\mathbf{J}$, $\mathbf{P}$ and is such that the remaining relation (2.1.9) is also satisfied.

We now set

$$\mathbf{l} \equiv \mathbf{J} - \mathbf{R} \times \mathbf{P} \tag{2.2.3}$$

and observe that it commutes with $\mathbf{P}$, $\mathbf{R}$, H, μ, and C_2. Thus, $\mathbf{l}$ may be identified with the internal angular momentum operator of the isolated molecule; it is a constant of motion which is also translation

invariant and unaffected by velocity transformations.† Furthermore, we see that

$$C_1 = (\mu \mathbf{l})^2 \tag{2.2.4}$$

The possible values of our operators depend, of course, on the particular dynamics given by the choice of H. For example, for a hydrogen atom without spin the values of $C_2/2\mu$ are given by $-13{\cdot}6$ eV$/n^2$ with $n = 1, 2, \ldots$. However, *the full problem involves the determination of all possible eigenvalues of* C_1 *and* C_2 for the system. In view of (2.1.10a–c), the parameters of the problem are chosen to be

$$\left.\begin{aligned} \mu \text{ (mass)} \\ E_0 = C_2/2\mu \text{ (internal energy)} \\ l(l+1) = C_1/\mu^2 \text{ (internal angular momentum)} \end{aligned}\right\} \tag{2.2.5}$$

The value of μ is the same for all states of the molecule and leads to a superselection rule among different molecules, *preventing the existence of unstable molecules*, as has been emphasized by Levy-Leblond (1963), with the limitation to free rather than composite particles.

The very existence of the parameter E_0 allows one to construct a mathematical structure of *stable isolated excited states* which are *not* the physical states. The physical states decay, but from the mathematical point of view this occurs only because one 'turns on' an external field. The mathematical separation of the object of study from the external word is possible in nonrelativistic quantum mechanics, but such a mathematical separation is excluded in relativistic theories, except for free particles. Thus, when Levy-Leblond (1963) calls E_0 an arbitrary parameter which can be set equal to zero he, like Hammermesh (1960), is forced to deal only with free elementary particles. In this framework [as well as in Voison's work (Voison, 1965a, b)] composites enter only through the tensor product of two representations of $\mathscr{G}$ which are then allowed to interact. Indeed, this kind of treatment is the only one permitted for $\mathscr{P}$, but a considerably simpler treatment is allowed for $\mathscr{G}$.

Actually the general representations constructed in detail by Voison (1965b) [using Mackey's method (Mackey, 1955)] and by Levy-Leblond (1963) [using little groups (Wigner, 1939)] also cover

† The system is invariant under the internal SU_2 group generated by $\mathbf{l}$ and hence under the group $(SU_2^{\text{int}} \times \mathscr{G})/Z_2$, as shown by Fleischman & Nagel (1966). Indeed this is true for a general I.R. of $\mathscr{G}$ and not merely for the molecule model.

the general case of a molecule. *If one wishes to describe all the states of an isolated molecule* (or more precisely, the direct sum of the I.R.'s of the associated Lie algebra), *all that one has to do is to describe the direct product by a single* μ *and many values of* C_1 *and* C_2. *It will be this direct sum structure that we later seek as the contracted limit of a relativistic composite particle with all its bound state levels.* Thus, the hydrogen atom replete with all its excited states (which has been excluded in the work of Newton & Wigner (1949) as a particle since it is not elementary) will now appear as a set of noninteracting *stable particles*, each with a mass operator μ, a position operator $\mathbf{R}$, an internal angular momentum operator $\mathbf{l}$, and an internal energy operator H_0. In other words, the contraction procedure provides us with a method of 'turning off the interaction' with the external world.

In order to elaborate on our ideas expressed above, we first recall that the analysis of free nonrelativistic particles has shown (among other things) that they possess only the attributes of a free relativistic particle, namely μ and l. *As a generalization, we now define a nonrelativistic composite particle as an entity whose states are described by a set of I.R.'s of* $\mathscr{G}$, *each with the same* μ, *but with a definite range of values*† *for* $C_1 = \mu^2 l(l+1)$ *and* $C_2 = 2\mu E_0$. Each composite particle is described by the operators of the Galilei group, with the obvious identification from our molecule discussion, i.e.,

$$\left.\begin{aligned} &\mu \equiv \text{Galilean mass operator} \\ &\mathbf{P} \equiv \text{momentum operator} \\ &\mathbf{R} \equiv \mu^{-1}(t\mathbf{P} - \mathbf{K}) \equiv \text{canonical position operator} \\ &\boldsymbol{\sigma} \equiv \mathbf{l} \equiv \mathbf{J} - \mathbf{R} \times \mathbf{P} \equiv \text{spin operator} \\ &H_0 \equiv H - \mathbf{P}^2/2\mu \equiv \text{internal energy operator} \end{aligned}\right\} \qquad (2.2.6)$$

The interactions of these particles with each other will be described via the tensor product space, forming an I.R. of $\mathscr{G}$, as was exemplified by our molecule discussion. In general, there will be scattering states as well as bound states. The two-particle problem has been treated in detail by Levy-Leblond (1963) and by Voison (1965b). They obtained the reduction of each I.R. of the two-particle system into the I.R.'s of each particle contained within it. The generalization to N-particle states is not difficult to visualize. Finally, we note that the interactions of a composite particle with an external field will not in

† The eigenvalues l are integers or half-integers (associated with the covering group of $\mathscr{G}$ or $\mathscr{P}$) and the eigenvalues of E_0 corresponding to bound state levels will have to form a discrete set, while scattering states lie in the continuum.

general yield Galilean invariant states, since the external field does not necessarily possess this symmetry.

3. *Relativistic Particles*

3.1. *Definition of the nonrelativistic H*

In Section 2.1 we described the contraction $\mathscr{P} \to \Gamma$ where Γ was the algebra of $\mathscr{G}$ without the nonrelativistic Hamiltonian H. The operator H was only defined for the free particle representations for which $C_2 = 2\mu H - \mathbf{P}^2 = 0$. We now present a prescription for defining a more general H such that C_2 is a nonvanishing operator, capable of taking on different eigenvalues for the limit states of the space S' which arises upon contraction from the relativistic state space S.

Our definition of H is motivated by the work of Foldy (1961), who developed a relativistic construct for a finite number of interacting particles, assuming an expansion of P_0 and M_{0j} in powers of c. All generators P_μ, $M_{\mu\nu}$ are to be explicitly constructed functions of the dynamical particle variables $\mathbf{r}^{(\alpha)}$, $\mathbf{p}^{(\alpha)}$, $\boldsymbol{\sigma}^{(\alpha)}$.

Foldy's expansion, with $H^{(0)}$ and $K_j^{(0)}$ taken to be the Galilean generators,† is

$$\begin{aligned} cP_0 &= \mu c^2 + H^{(0)} + H^{(1)} + \cdots \\ M_{0j}/c &= K_j^{(0)} + K_j^{(1)} + \cdots \end{aligned} \tag{3.1.1}$$

The operators μ, $H^{(n)}$, and $K_j^{(n)}$ are thus defined to exist on the relativistic space S of states and are constructed from the particle dynamical variables. In contrast, in our construct *the only additional operator assumed to be defined on S, shall be* μ.

Foldy's result (Foldy, 1961, equation 75) is

$$P_0 = c^{-1}(h^2 + c^2\mathbf{P}^2)^{1/2} \tag{3.1.2}$$

where

$$h = \mu c^2 + h^{(0)} + h^{(1)} + \cdots \tag{3.1.3}$$

with

$$h^{(0)} = U^{(0)} + \sum_\alpha \boldsymbol{\pi}^{(\alpha)2}/2m^{(\alpha)} \tag{3.1.3a}$$

$$h^{(1)} = -(2\mu c^2)^{-1} h^{(0)2} + W^{(1)} \tag{3.1.3b}$$

† We note here that Foldy does not make the assumptions concerning $\mathbf{K}$ as we did, compare our paragraph following equation (2.2.1a). Hence, in order to obtain $\mathbf{K} = t\mathbf{P} - \mu\mathbf{R}$, he relies on a questionable unitary transformation to a special representation, see Foldy (1961, p. 281). His reasoning assumes more generally that for the interacting Galilean system, $\mathbf{K} \to \mathbf{K} + \mathbf{V}$ and $H \to H + U$, as suggested by Dirac (1949). Foldy notes that while one can satisfy the commutation relations of $\mathscr{G}$ for $U \neq 0$ with $\mathbf{V} = 0$, this is not the case in respect to $\mathscr{P}$. Hence he declines to make the assumption $\mathbf{V} = 0$ *ab initio* even for $\mathscr{G}$.

Here $U^{(0)}$ is the Galilean potential energy operator, and $W^{(1)}$ is a rotationally invariant function of the internal variables† $m^{(\alpha)}$, $\boldsymbol{\sigma}^{(\alpha)}$ and

$$\begin{aligned} \boldsymbol{\rho}^{(\alpha)} &\equiv \mathbf{R} - \mathbf{r}^{(\alpha)} \\ \boldsymbol{\pi}^{(\alpha)} &\equiv \mathbf{p}^{(\alpha)} - \mu^{-1} m^{(\alpha)} \mathbf{P} \end{aligned} \tag{3.1.3c}$$

Comparing (3.1.2) and (2.1.2) we see that Foldy's h is related to the relativistic mass M by the relation

$$h = Mc^2 \tag{3.1.4}$$

Hence, (3.1.3) can be rewritten as

$$(M - \mu)\, c^2 = h^{(0)} + h^{(1)} + \cdots \tag{3.1.5}$$

The $h^{(0)}$ of Foldy's expansion is independent of c; in our viewpoint, *it is precisely the internal energy operator H_0 defined by Foldy to act on the relativistic state space S.* In the limit when $c \to \infty$,

$$(M - \mu)\, c^2 \to H_0 \tag{3.1.6}$$

For bound states, H_0 has negative eigenvalues, which describe the binding energy of the system. Hence *the Galilean mass μ is greater than the relativistic mass M.* We recall that in our framework the c-independent operator μ is defined by equation (2.1.4). *We then take equation* (3.1.6) *as our definition of the nonrelativistic internal energy H_0, and we require that H_0 commute with all operators of $\Gamma = \{\mu, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$.* It then follows that H_0 *will also commute with the nonrelativistic Hamiltonian H defined by*

$$H \equiv H_0 + \mathbf{P}^2/2\mu \tag{3.1.7}$$

Hence H_0 is a Casimir operator of $\mathscr{G}$, with the eigenvalues $E_0 = C_2/2\mu$.

It is readily seen that the relativistic operator that goes to H as $c \to \infty$ is‡

$$T + (M - \mu)\, c^2 = (P_0 c - Mc^2) + (M - \mu)\, c^2 \tag{3.1.8a}$$

Thus we see that

$$P_0 c - \mu c^2 \to H \tag{3.1.8b}$$

With μ being the Galilean restmass and with (3.1.6), these equations give a pictorial interpretation of H.

† The *internal coordinates* $\boldsymbol{\rho}^{(\alpha)}$ and $\boldsymbol{\pi}^{(\alpha)}$ defined by (3.1.3c) commute with $\mathbf{P}$ and $\mathbf{R}$. They obey the relations $\sum m^{(\alpha)} \boldsymbol{\rho}^{(\alpha)} = 0 = \sum \boldsymbol{\pi}^{(\alpha)}$. For a further discussion of internal variables see Fleischman, O. (1967), *Boulder Summer Institute of Theoretical Physics*, to be published.

‡ The definition of T is given by (2.1.8).

The results of Hammermesh (1960) and Levy-Leblond (1963) depend on the circumstance that $\mathbf{P}^2/2\mu$ (which, in that framework, is the Hamiltonian) obeys the commutation relations of $\mathscr{G}$. We now show that *our* H also has this necessary property. First we observe that H [as defined by (3.1.7)] commutes† with all the operators of $\mathscr{G}$ except $\mathbf{K}$. The commutator $[H, K_j]$ is derived by noting that from the third relation in (2.1.6) it follows that

$$[(2\mu)^{-1}\mathbf{P}^2, K_j] = (2\mu)^{-1}(-2i\mu P_j) = -iP_j$$

In summary, we then have

$$\left.\begin{aligned} [H,\mu] = [H,\mathbf{P}] = [H,\mathbf{J}] = 0 \\ [H,K_j] = -iP_j \end{aligned}\right\} \tag{3.1.9}$$

These relations justify calling H the Hamiltonian operator in the I.R. of $\mathscr{G}$ that describes the nonrelativistic stationary composite particle states. In passing we note that the correct commutation relations (3.1.9) follow because $\mathbf{P}^2/2\mu$ satisfies them and because H does not occur as a commutator of $\mathscr{G}$. This last circumstance is a remarkable distinguishing feature of $\mathscr{G}$ as contrasted with $\mathscr{P}$.

We shall explore in a subsequent paper the commutation structure of μ, P_ν, and $M_{\nu\lambda}$. Presently, we shall give an application of the construct which we developed above.

3.2. *The nonrelativistic limit of approximate* SU_3

In a previous paper (Fleischman & Roman, 1967) we studied the nonrelativistic limit $c \to \infty$ of the SU_3 commutation relations satisfied by P_0, given the Okubo and Gell-Mann prescription that‡

$$P_0 = aY' \tag{3.2.1}$$

Thus, with $X_\pm = U_\pm$ or $V_\pm$ of SU_3, we have

$$[P_0, X_\pm] = \pm aX_\pm' \tag{3.2.2}$$

We assumed in a previous paper (Fleischman & Roman, 1967) that a was c-dependent but Y', $U_\pm'$, $V_\pm'$, $\mathbf{I}'$ and Y, $U_\pm$, $V_\pm$, $\mathbf{I}$ did not depend on c. We then deduced that $[\mu, X_\pm] = 0$, while H_0 could satisfy $[H_0, X_\pm] = \pm a' X_\pm'$ and hence have *eigenvalues* obeying the GMO mass splitting formula. In that calculation we assumed (a) that P_μ was the sum of two vectors $P_\mu^{(1)}$ and $P_\mu^{(2)}$, (b) that $[P_\mu^{(1)}, P_\nu^{(1)}] = 0$, and (c) that $P_\mu^{(1)}$ and $[P_\mu^{(1)}, P_\nu]$ vanished when $c \to \infty$. We then defined

† Note that H_0 in (3.1.7) has been shown to be a Casimir operator.

‡ Regarding notation, see Fleischman & Roman (1967).

$H_0 \equiv -cP_0^{(1)}$. We also confined ourselves to the rest frame states with $\mathbf{P} = 0$.

In our present framework, we replace (3.2.1) by

$$M = aY' \tag{3.2.3}$$

and hence have

$$[M, X_\pm] = \pm aX_\pm' \tag{3.2.4}$$

Since $M \to \mu$, we get

$$[\mu, X_\pm] = \pm X_\pm' \lim_{c\to\infty} a \equiv \pm bX_\pm' \tag{3.2.5}$$

Thus, from (3.1.6), (3.2.3) and (3.2.4),

$$[H_0, X_\pm] = \pm c^2(a-b)\, X_\pm' \tag{3.2.6}$$

which must remain finite. Consequently, $a - b$ is of order less than c^{-2}.

If we choose a independent of c, then we have the same value of H_0 with μ (and also M) satisfying the mass splitting formula. Thus, we are led to a set of independent nonrelativistic particles with the same internal energy and different masses. If one does not consider the relationship of this multiplet to other multiplets, then H_0 can be regarded as a free parameter [compare Levy-Leblond (1963)] which can be set equal to the zero of the energy scale so that we have a set of free particles with masses obeying the GMO formula.

If, however, $\lim a \equiv b = 0$ when $c \to \infty$, then by (3.2.5), μ is an SU_3 scalar. If, furthermore, $c^2 a$ stays finite when $c \to \infty$, then by (3.2.6) H_0 will give rise to the GMO formula.

Thus, our new prescription (3.2.3), together with the old assumption that $X_\pm$ and $X_\pm'$ is independent of c, and with the only further assumption that

$$a = 0(c^{-2}) \tag{3.2.7}$$

reproduces our previous results (Fleischman & Roman, 1967), which were

$$\left.\begin{aligned} \lim c^2 a &= a' < \infty \\ [\mu, X_\pm] &= 0 \\ [H_0, X_\pm] &= \pm a' X_\pm' \end{aligned}\right\} \tag{3.2.8}$$

All hadrons with a given common spin and parity have as their nonrelativistic limit a set of I.R.'s of $\mathscr{G}$ which look like those of an isolated molecule (or atom): they have common mass μ (and common spin) and differing internal energy eigenvalues

$$E_0(t, y) = A + By + C[t(t+1) - y^2/4] \tag{3.2.9}$$

In the framework of the theory as established in Section 2, no specific assumptions as were used in our previous paper (Fleischman & Roman, 1967) are needed.

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